On the stability of a spinning top containing liquid

By K. STEWARTSON
Department of Mathematics, The Durham Colleges in the University of Durham

WITH AN APPENDIX BY G. N. WARD

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The stability of a heavy top, containing a cylindrical cavity partly full of liquid, for small displacements from the sleeping position is studied. It is shown theoretically that instability can occur when any one of the periods of free oscillation of the liquid, which are doubly infinite in number, is sufficiently near to the period of nutation of the empty top. In experiments carried out by Prof. Ward, only the two principal instabilities could be distinguished.

1. Introduction

Consider an axisymmetrical solid top of mass M rotating about a fixed point O of its axis at a distance d from its centre of gravity G. It is well known that the top is stable in a sleeping position with G vertically above O provided that

$$L^2\Omega^2 > 4MgdT, \tag{1.1}$$

where L, T are the moments of inertia of the top about longitudinal and transverse axes through O and Ω is the angular velocity of the top about OG.

If, however, the top has a cavity containing fluid, this simple criterion is no longer adequate. It was in fact shown by Kelvin (1877) that the top could be made unstable by choosing a suitable cavity. He exhibited a thin-walled spheroidal top full of liquid. Originally its shape was just oblate and it was stable in the sleeping position if spun fast enough. However, on hammering it into a slightly prolate form and releasing it from the sleeping position, it became violently unstable to the extent of damaging the bearings on which it was spinning. The mathematical explanation of this phenomemon was given by Greenhill (1880). He assumed that the fluid inside the top was rotating with the casing as if solid to begin with, and studied the perturbations in the motion of the fluid consequent to a small disturbance being applied to the top. The perturbed motion produces pressure variations on the inner surface of the casing leading to a couple which may augment the destabilizing couple due to gravity. However, the new couple is proportional to Ω^2, and so it is possible that no matter how rapidly the top is rotated it will still be unstable. It is thus a different kind of instability from that occurring with a solid top, which can always be removed by choosing a sufficiently high angular velocity. In particular, Greenhill was able to show that when the casing has negligible mass and is pivoted at the centre of the cavity, the top is unstable if $a < c < 3a$, where $2a$, $2a$, $2c$ are the lengths of the principal axes of the spheroidal cavity. As the mass of the casing is increased, the unstable range of c is narrowed until in the limit body instability occurs at only one value of c given by

$$\tau_{\mathrm{nu}} = \frac{c^2 - a^2}{c^2 + a^2}, \tag{1.2}$$

where $2\pi/\Omega\tau_{\mathrm{nu}}$ is the period of nutation of the top when the cavity is empty.

Similar results have been obtained by Hough (1898) when the cavity is ellipsoidal with one principal axis along the axis of symmetry of the empty top. The restriction that the centre of the cavity coincide with the centre of rotation, which was assumed by all three writers mentioned so far, has been removed in unpublished work by S. N. Barua, E. A. Milne and the present author. The modification necessary is not difficult and does not affect the general nature of the theory.

In practice, however, it is difficult to ensure that the cavity in the top is completely full of liquid; accordingly, it is of interest to study the top's stability when the cavity is only partly full. The liquid now has a free surface whose shape depends on the relative strength of the gravitational and centrifugal forces. In this paper we shall assume that the top is rotating sufficiently rapidly that the gravitational forces may be neglected, so that the free surface is taken to be a cylinder parallel to the axis of symmetry. The condition to be satisfied is $a^2\Omega^2 \gg gc$, where a, c are now representative lengths in the cavity across and along the axis of symmetry.

The gain in simplicity obtained by choosing a spheroidal cavity, which is so clear in Greenhill's work, is now lost. Instead we shall suppose that the cavity is a finite cylinder, whereupon the boundary conditions take on a particularly convenient form and lead to a comparatively simple solution. This problem has also been considered by Narimanov (1957) who assumes, however, that the cavity is nearly empty and neglects the variation of the velocities with r, the distance from the axis. No results comparable with those of the present paper were given, but it is likely that Narimanov's assumption is very restrictive.

It is assumed throughout the paper that the liquid is compressible and inviscid, and is initially rotating with the top as if solid. Strictly speaking, an inviscid liquid cannot be given vorticity by any motion of its boundaries, and it may be argued that for consistency the liquid should be assumed to be at rest initially inside the cavity. Such an approach has been used by Chetayev (1957). However, no real liquid is completely inviscid, and so any liquid will be dragged round to some extent by the rotating casing. From considerations of diffusivity, the liquid in the top can be expected to rotate substantially as if solid after a time of the order of a^2/ν, where a is a representative length and ν the kinematic viscosity of the liquid. Chetayev's theory may then be thought of as applying at the initial stages of the motion of the top. As the liquid slowly speeds up, acquiring vorticity from the motion of the casing, the criteria for instability will change, tending ultimately to those of the present paper.

The stability equation for a top with a spheroidal cavity completely full of liquid is particularly simple, leading to a single range of values of c/a in which the top is unstable. When the cavity is a cylinder of length $2c$ and radius a, with or without an air space, it is found that the stability equation is much more complicated, and leads to an enumerable infinity of ranges of values of c/a in any of which the top is unstable. The total length of all these ranges is probably of the same order as the length of the single range occurring with a spheroidal cavity; the fragmentation means that instability is liable to occur unexpectedly, on making apparently trivial changes in the properties of the top.

The reason is that in general the liquid in the cavity has an infinite number of normal modes, each having a different period of oscillation. The motion of the liquid in a normal mode gives rise to a fluctuating couple on the casing of the top which acts as a disturbing force on the motion of the solid part of the top. There is also a reverse mechanism in which the motion of the casing induces a motion of the liquid in the cavity. Instability occurs when the period of nutation of the casing is sufficiently near to any one of the periods of normal modes of the liquid. It turns out that the period of precession is not directly associated with instability. The theory of the cylindrical cavity is an example of this argument. The completely filled spheroidal cavity is a special case, however, because although there are an infinite number of normal modes of oscillation of the liquid, in only one is there induced a resultant couple on the casing. Further, the motion of the casing induces only one normal mode of oscillation of the liquid. Hence, instead of an infinite number of resonances, there is only one; but, on the other hand, it has a broader band-width than any of the corresponding resonances for another shape of cavity. If the spheroidal cavity is not completely filled, these special considerations no longer apply, and presumably fragmentation of the band-width again occurs although the problem is too difficult to solve completely at present.

Experiments to check the theory of the present paper were carried out by Prof. Ward who discusses them in the Appendix. He was able to confirm the presence of the principal mode of resonance, but found that its band-width was much larger than predicted by the theory. One other mode of resonance was detected, but the apparatus apparently was not sufficiently sensitive to detect any more of the weak secondary modes.

2. Equations of motion and boundary conditions

Suppose that in dynamic equilibrium the top is rotating about a fixed point O on its axis of symmetry; it has an angular velocity Ω about its axis of symmetry which is vertical and the centre of gravity of the top is above O. The top contains a cylindrical cavity of radius a and length $2c$, whose axis is the axis of symmetry and whose centre is at a distance h from O. It is partly filled with a volume $2\pi c(a^2-b^2)$ of inviscid liquid of density ρ which is rotating with the top as if solid. Thus, when the conclusions of the theory are compared with experiment, we must assume that before being disturbed the top has been rotating sufficiently long for the liquid to have taken up its angular velocity, but that in the subsequent disturbed motion viscosity may be neglected. Further, it is assumed that gravitational forces in the liquid may be neglected in comparison with the centrifugal forces. The gravitational forces on the solid casing of the top are not negligible, however, and so the assumption is equivalent to saying that the mass of liquid is small compared with the mass of the casing. There is no formal difficulty in including the effect of gravitational forces provided that the inner surface of the liquid is effectively a cylinder concentric with the cavity, i.e. $a^2\Omega^2 \gg gc$, where g is gravity.

It is convenient to introduce two systems of rotating orthogonal triads each with origin O: (*a*) the triad $Oxyz$ in which Oz is vertically upwards and Ox, Oy rotate about it with angular velocity Ω; (*b*) the triad $Ox'y'z'$ in which Oz' lies along the axis of symmetry and Ox', Oy' rotate about it with angular velocity Ω. The components of a property with respect to $Ox'y'z'$ are distinguished from the components of the same property with respect to $Oxyz$ by primes.

Let there be a small disturbance in the motion of the top in which Oz' has at any time direction cosines (l, m, n) with respect to $Oxyz$. Since in equilibrium $l = m = 0$, $n = 1$, and $l^2+m^2+n^2 = 1$, it follows that in the disturbed motion $n-1$ is of second order. Let the velocity of the liquid at (x, y, z) be (u, v, w), where again u, v, w are small. Then, neglecting the gravitational body force in the liquid and writing

$$p = \rho P + \tfrac{1}{2}\rho\Omega^2(x^2+y^2), \tag{2.1}$$

where p is the pressure, we have, following Proudman (1916),

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{\partial P}{\partial x}, \qquad \frac{\partial v}{\partial t} + 2\Omega u = -\frac{\partial P}{\partial y},$$
$$\frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.2}$$

Here and subsequently we neglect all second-order quantities.

In order to study the stability of the equilibrium position, we now assume that l, m, u, v, w, P are each proportional to e^{st}, where s is a constant. The coefficient of e^{st} in each case is denoted by a suffix s, so that for example

$$u(x, y, z, t) = u_s(x, y, z, s)\, e^{st}. \tag{2.3}$$

The method is then to determine the motion of the liquid due to the perturbation of the casing and the reaction of the liquid on the motion of the casing. It is found that the assumption is only justified if s satisfies a certain transcendental equation, and the condition of stability is that no roots of this equation should have positive real parts.

In terms of s, equations (2.2) reduce to

$$\frac{\partial^2 P_s}{\partial x^2} + \frac{\partial^2 P_s}{\partial y^2} = \alpha^2 \frac{\partial^2 P_s}{\partial z^2}, \quad \text{where} \quad \alpha^2 = -\frac{s^2+4\Omega^2}{s^2}, \tag{2.4}$$

$$\alpha^2 s^2 u_s = s\frac{\partial P_s}{\partial x} + 2\Omega\frac{\partial P_s}{\partial y}, \quad \alpha^2 s^2 v_s = -2\Omega\frac{\partial P_s}{\partial x} + s\frac{\partial P_s}{\partial y}, \tag{2.5}$$
$$s w_s = -\frac{\partial P_s}{\partial z}.$$

The condition on any surface $F(x, y, z, t) = 0$ bounding the liquid is that

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \quad \text{when} \quad F = 0, \tag{2.6}$$

since it always consists of the same particles of liquid. Further, the coordinates of any point satisfy

$$z' = z + lx + my. \tag{2.7}$$

The two formulae (2.5) and (2.7) are now applied to the plane and curved boundaries of the liquid. First, since the equations to the plane faces of the cavities are $z' = h \pm c$, it follows that

$$w_s = -s(l_s x + m_s y) \quad \text{when} \quad z' = h \pm c. \tag{2.8}$$

Secondly, the curved part of the cavity is distant a from Oz'; and hence, for any point on it,

$$a^2 + z'^2 = x^2 + y^2 + z^2,$$

i.e.

$$x^2 + y^2 - 2lxz - 2myz - a^2 = 0. \tag{2.9}$$

The boundary condition is therefore

$$u_s x + v_s y = sz(l_s x + m_s y) \quad \text{when} \quad x^2 + y^2 = a^2. \tag{2.10}$$

Thirdly, the pressure is constant on the free boundary, which in equilibrium is distant b from Oz'. Hence, from (2.1), the boundary condition is

$$sP_s + \Omega^2(u_s x + v_s y) = 0 \quad \text{when} \quad x^2 + y^2 = b^2. \tag{2.11}$$

The inner boundary is not exactly a cylinder, and if necessary its equation may be found in terms of P. Let the distance of any point on it from Oz' be $b + \eta'$. Then

$$(b + \eta')^2 + z'^2 = x^2 + y^2 + z^2,$$

whence, from (2.6) and (2.7),

$$b\Omega^2 \eta' = -P - \Omega^2(lx + my), \tag{2.12}$$

the right-hand side being evaluated at $x^2 + y^2 = b^2$.

These boundary conditions may be expressed in terms of a new variable Q_s, defined by

$$Q_s = P_s - s^2 z(l_s x + m_s y), \tag{2.13}$$

and which satisfies the same equation as P_s. We have, writing

$$r\cos\theta = x, \quad r\sin\theta = y,$$

$$\frac{\partial Q_s}{\partial z} = 0 \tag{2.14}$$

when $z = h \pm c$,

$$sr\frac{\partial Q_s}{\partial r} + 2\Omega\frac{\partial Q_s}{\partial \theta} = -2sza(l_s\cos\theta + m_s\sin\theta)(s^2 + 2\Omega^2) + 2\Omega s^2 za(l_s\sin\theta - m_s\cos\theta) \tag{2.15}$$

when $r = a$, and

$$sr\frac{\partial Q_s}{\partial r} + 2\Omega\frac{\partial Q_s}{\partial \theta} - s(s^2 + 4\Omega^2)Q_s = s^3 zb(s^2 + 3\Omega^2)(l_s\cos\theta + m_s\sin\theta) + 2\Omega^2 s^3 zb(l_s\sin\theta - m_s\cos\theta) \tag{2.16}$$

when $r = b$.

3. The motion of the liquid

The boundary conditions suggest that we look for solutions of the differential equation (2.4) satisfied by Q_s which are proportional to $\cos\theta$, $\sin\theta$. The most general solution of this form is

$$Q_s = \sum_k C_k\{A_k(r)\cos\theta + B_k(r)\sin\theta\}\cos k(z-h+c), \tag{3.1}$$

where the summation is over a range of values of k to be determined, C_k are constants introduced for convenience, and $A_k(r)$, $B_k(r)$ are Bessel's functions of order one with argument αkr. There are other solutions of (2.4), but none of them can depend on l_s, m_s, nor can they contribute to the couple exerted by the liquid on the casing of the top.

From (2.14) it follows that

$$k = 0 \quad \text{or} \quad k = \frac{\pi}{2c}(2j+1), \tag{3.2}$$

where j is a positive integer including zero.

We now expand z as a Fourier cosine series in the range $h-c \leqslant z \leqslant h+c$, obtaining

$$z = h - \frac{8c}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\cos\left\{\frac{\pi}{2c}(2j+1)(z-h+c)\right\} = \sum_k C_k \cos k(z-h+c) \tag{3.3}$$

on setting $C_0 = h$ and $C_k = -2(ck^2)^{-1}$ when $k \neq 0$. Substituting (3.1) and (3.3) into (2.15), we find that when $r = a$

$$sa\frac{dA_k}{dr} + 2\Omega\frac{dB_k}{dr} = -2as(s^2+2\Omega^2)\,l_s - 2as^2\Omega m_s, \tag{3.4}$$

$$-2\Omega A_k + sa\frac{dB_k}{dr} = 2as^2\Omega l_s - 2as(s^2+2\Omega^2)\,m_s, \tag{3.5}$$

i.e.
$$\left(sa\frac{d}{dr} - 2i\Omega\right)(A_k+iB_k) = -2as(s+i\Omega)(s-2i\Omega)(l_s+im_s). \tag{3.6}$$

Similarly, when $r = b$,

$$\left(sb\Omega^2\frac{d}{dr} - 2i\Omega^3 - s(s^2+4\Omega^2)\right)(A_k+iB_k) = -bs^2(s-2i\Omega)(s+i\Omega)^2(l_s+im_s). \tag{3.7}$$

First consider the special case when $k = 0$. Here

$$A_0 + iB_0 = (l_s+im_s)(X_0 r + Z_0/r), \tag{3.8}$$

where X_0, Z_0 are independent of r. On substituting into (3.6) and (3.7), it is found, after some algebra, that

$$X_0 a + Z_0/a = -2as(s+i\Omega) + \frac{2s^2ab^2(s+i\Omega)^2}{b^2(s+i\Omega)^2 + a^2(s^2+\Omega^2-2i\Omega s)}. \tag{3.9}$$

This is the only combination of X_0 and Z_0 needed in the argument.

Next consider the general case $k \neq 0$. Write

$$\frac{A_k+iB_k}{l_s+im_s} = X_k J_1(\alpha kr) + Z_k Y_1(\alpha kr) = \mathscr{C}_1(\alpha kr), \tag{3.10}$$

where X_k, Z_k are constants and J, Y are Bessel's functions of the first and second kind with real argument. Let us also write

$$X_k J_0(\alpha kr) + Z_k Y_0(\alpha kr) = \mathscr{C}_0(\alpha kr). \tag{3.11}$$

Then the boundary conditions become

$$\left.\begin{aligned} \alpha sb\Omega^2\mathscr{C}_0(\alpha kb) - (s+2i\Omega)(s^2-2i\Omega s+\Omega^2)\mathscr{C}_1(\alpha kb) &= bs^2(s-2i\Omega)(s+i\Omega)^2, \\ \alpha sa\mathscr{C}_0(\alpha ka) - (s+2i\Omega)\mathscr{C}_1(\alpha ka) &= -2sa(s+i\Omega)(s-2i\Omega). \end{aligned}\right\} \tag{3.12}$$

4. The couple on the casing

In this section we calculate the couple on the casing exerted by the motion of the liquid described in the previous section. Contributions to the couple come from the curved surface and the plane parts of the cavity, and we shall consider them separately. The pressure in the liquid is

$$p = \tfrac{1}{2}\rho\Omega^2(x^2+y^2) - \tfrac{1}{2}\rho\Omega^2 b^2 + p_0 + \rho P, \tag{4.1}$$

where p_0 is the constant pressure in the air space ($r' < b$).

Let the components of the couple exerted on the curved surface be (E_c, F_c, G_c) referred to $Oxyz$ and (E'_c, F'_c, G'_c) referred to $Ox'y'z'$. Then $G'_c = 0$; and, since all other components are first order,

$$E_c = E'_c, \quad F_c = F'_c. \tag{4.2}$$

Now
$$E'_c + iF'_c = \frac{i}{a}\iint z'(x'+iy')\,p\,dS', \tag{4.3}$$

where the integral is taken over the curved surface $r' = a$, $|z'-h| \leqslant c$. From (4.1) and (2.9), $p = \rho P + \rho\Omega^2(lx'z'+my'z') + \text{const.}$ on this surface, and therefore

$$E'_c + iF'_c = \frac{i\rho}{a}\iint \{P + \Omega^2(lx'z'+my'z')\}\, z'(x'+iy')\,dS'. \tag{4.4}$$

Substituting from (3.1), (3.9), (3.10) and dropping the primes, we get

$$\frac{(E_c+iF_c)_s}{l_s+im_s} = \tfrac{2}{3}i\rho\pi a^2c^3(s^2+\Omega^2) - 2i\pi\rho ca^2h^2(s+i\Omega)^2 + i\pi\rho ac\sum_{j=0}^{\infty} C_k^2\mathscr{C}_1(\alpha ka) - \frac{4i\pi ca^2h^2(s+i\Omega)^2(s^2-2i\Omega s+\Omega^2)(a^2-b^2)}{b^2(s+i\Omega)^2+a^2(s^2-2i\Omega s+\Omega^2)}. \tag{4.5}$$

The couple exerted by the liquid on the plane faces of the cavity may be found on the assumption that the liquid extends from $r' = b$ to $r' = a$. This is not strictly correct, of course, for the inner boundary of the liquid is $r' = b+\eta'$, where η' is defined in (2.12). However, since the pressure perturbation in $b < r' < b+\eta'$ is first-order, the couple exerted on that part of the plane faces is of second-order and negligible.

Relative to $Ox'y'z'$, let the components of the couple exerted on the plane face $z' = h+c$ be $(E'_+, F'_+, 0)$. Then

$$E'_+ + iF'_+ = -i\rho\int_0^{2\pi} d\theta' \int_b^a (x'+iy')\,[P + \tfrac{1}{2}\Omega^2(r'^2 + 2lx'z' + 2my'z')]\,dr'. \tag{4.6}$$

Again the primes may be dropped. After some reduction it is found that

$$\frac{(E_+ + iF_+)_s}{l_s + im_s} = i\pi\rho \sum_{j=0}^{\infty} C_k \int_b^a r^2 \mathscr{C}_1(\alpha k r)\, dr - \tfrac{1}{4} i\rho\pi(a^4 - b^4)(h+c)(s^2+\Omega^2) + i\pi\rho h \int_b^a r^2(X_0 r + Z_0/r)\, dr. \quad (4.7)$$

Similarly, if the couple exerted by the liquid on the plane face $z' = h - c$ is $(E'_-, F'_-, 0)$, then

$$\frac{(E_- + iF_-)_s}{l_s + im_s} = i\pi\rho \sum_{n=0}^{\infty} C_k \int_b^a r^2 \mathscr{C}_1(\alpha k r)\, dr - \tfrac{1}{4} i\rho\pi(a^4 - b^4)(h-c)(s^2+\Omega^2) - i\pi\rho h \int_b^a r^2(X_0 r + Z_0/r)\, dr. \quad (4.8)$$

Now
$$\int_b^a r^2 \mathscr{C}_1(\alpha k r)\, dr = \frac{1}{\alpha^2 k^2}\left[2r\mathscr{C}_1(\alpha k r) - \alpha k r^2 \mathscr{C}_0(\alpha k r)\right]_b^a, \quad (4.9)$$

and we may use the boundary conditions (3.4) to eliminate $\mathscr{C}_0(\alpha k b)$ and $\mathscr{C}_0(\alpha k a)$.

Hence finally, if F, G are the components of the couple about Ox, Oy, we have, adding (4.5), (4.7) and (4.8),

$$\begin{aligned}\frac{F_s + iG_s}{l_s + im_s} = {}& \tfrac{2}{3} i\pi\rho a^2 c^3 \frac{(s+i\Omega)(3s - 2i\Omega)}{(s+2i\Omega)} + \tfrac{2}{3} i\pi\rho b^2 c^3 \frac{s^3(s+i\Omega)^2}{\Omega^2(s+2i\Omega)} \\ & - \tfrac{1}{2} i\pi\rho c(a^4 - b^4)(s^2+\Omega^2) - \frac{2i\pi\rho c a^2 h^2 (a^2 - b^2)(s+i\Omega)(s^2 - 2i\Omega s + \Omega^2)}{b^2(s+i\Omega)^2 + a^2(s^2 - 2i\Omega s + \Omega^2)} \\ & + \frac{2i\pi\rho c a(s+i\Omega)}{s+2i\Omega} \sum_{j=0}^{\infty} C_k^2 \mathscr{C}_1(\alpha k a) + \frac{i\pi\rho c b s(s+i\Omega)^2}{\Omega^2(s+2i\Omega)} \sum_{j=0}^{\infty} C_k^2 \mathscr{C}_1(\alpha k b). \quad (4.10)\end{aligned}$$

A partial check on the validity of this rather complicated expression is obtained by setting $b = a$. Then

$$\mathscr{C}_1(\alpha k a) = \mathscr{C}_1(\alpha k b) = -a(s^2 + \Omega^2) \quad (4.11)$$

from (3.12). On substituting (4.11) into (4.10) and noting that

$$\sum_{j=0}^{\infty} C_k^2 = \tfrac{2}{3} c^2,$$

it follows that $F_s + iG_s = 0$, as would be expected because the cavity is empty.

5. The motion of the top

Let H_x, H_y be the moments, about Ox, Oy respectively, of the rate of change of the momentum of the casing of the top, and let T, T, L be the principal moments of inertia of the casing about O. Then from dynamical considerations,

$$\left.\begin{aligned} H_x &= T(\ddot{m} + 2\Omega\dot{l} - \Omega^2 m) - \Omega L(\dot{l} - \Omega m), \\ H_y &= T(\ddot{l} - 2\Omega\dot{m} - \Omega^2 l) + \Omega L(\dot{m} + \Omega l). \end{aligned}\right\} \quad (5.1)$$

The disturbing couple consists of two parts. First, there is a contribution from the liquid in the cavity which has been calculated in the preceding section. Secondly, there is a contribution from the gravitational forces on the casing, their effect on the liquid being neglected. The equations of motion of the casing are therefore

$$H_x = E - Mg\,dm, \quad H_y = F + Mg\,dl, \tag{5.2}$$

where M is the mass of the casing and d the distance of its centre of gravity from O. These equations may be combined to give

$$(l_s + im_s)\,[T(s+i\Omega)^2 - i\Omega L(s+i\Omega)] = \frac{\Omega^2 L^2}{4T}\beta(l_s + im_s) - i(E_s + iF_s), \tag{5.3}$$

where for convenience we have set

$$4Mg\,dT = L^2\Omega^2\beta. \tag{5.4}$$

It is noted that the condition for the stability of the empty top is $\beta < 1$. On substituting for $E_s + iF_s$ from (4.10), $l_s + im_s$ cancels throughout and we obtain an equation for s in terms of the physical properties of the top. The equation must be satisfied if the assumption concerning the nature of the motion made in §2 (i.e. $l, m \propto e^{st}$), is to be justified. Corresponding to the roots of this equation are the normal modes of the hydrodynamic-dynamic system, which is stable therefore only if in none of the roots of the equation is $\mathscr{R}\{s\} > 0$. On writing $s = i\Omega(1+\tau)$, the equation for τ becomes

$$\begin{aligned} T\tau^2 - L\tau + \frac{L}{4\tau}\beta = &-\tfrac{1}{2}\pi\rho c\tau(\tau-2)(a^4-b^4) - \tfrac{2}{3}\rho\pi b^2c^3\frac{\tau^2(\tau-1)}{\tau+1} \\ &+ \tfrac{2}{3}\pi\rho a^2c^3\frac{\tau^2(3\tau-5)}{\tau+1} - \frac{2\pi\rho ca^2h^2(a^2-b^2)\,\tau^2(\tau^2-4\tau+2)}{a^2(\tau^2-4\tau+2)+b^2\tau^2} \\ &- \frac{\pi\rho c}{\Omega^2(1+\tau)}\sum_{j=0}^{\infty} C_k^2\{2a\tau\mathscr{C}_1(\alpha ka) + \tau^2(1-\tau)\,b\mathscr{C}_1(\alpha kb)\}, \end{aligned} \tag{5.5}$$

where $\alpha^2 = (3-\tau)(1+\tau)/(1-\tau)^2$, and k, C_k, are defined in (3.2) and (3.3).

Further, in (5.5),

$$\mathscr{C}_1(\alpha kr) = X_k J_1(\alpha kr) + Z_k Y_1(\alpha kr), \tag{5.6}$$

where from (3.12)

$$\left.\begin{aligned} &X_k\{\alpha ka(1-\tau)J_0(\alpha ka) + (1+\tau)J_1(\alpha ka)\} \\ &\quad + Z_k\{\alpha ka(1-\tau)Y_0(\alpha ka) + (1+\tau)Y_1(\alpha ka)\} = -2a\Omega^2(3-\tau)(1-\tau), \\ &X_k\{\alpha kb(1-\tau)J_0(\alpha kb) - (1+\tau)(2-4\tau+\tau^2)J_1(\alpha kb)\} \\ &\quad + Z_k\{\alpha kb(1-\tau)Y_0(\alpha kb) - (1+\tau)(2-4\tau+\tau^2)Y_1(\alpha kb)\} \\ &\quad = b\Omega^2\tau^2(3-\tau)(1-\tau)^2. \end{aligned}\right\} \tag{5.7}$$

It is noted that (5.5) is a real equation for τ, and hence the roots are either real or are complex conjugates. The condition for stability is therefore that the roots are all real. Further, $\mathscr{C}_1(\alpha kr)$ is regular *qua* function of τ, except at a discrete set of poles which we can expect to be real. Otherwise the motion of the liquid would be unstable if the casing were fixed. In numerical calculations by Dr D. C. Gilles of the Scientific Computing Service, the poles of X_k, Z_k found were all real.

A complete discussion of (5.5) would be exceedingly complicated in view of the number of parameters, and here we shall restrict attention only to the stability of a heavy top with a small cavity. In fact a system of this kind is necessary to justify the neglect of the gravitational forces on the liquid but not on the casing. In this special case the right-hand side of (5.5) is small except at the poles of X_k, Z_k and when $b^2\tau^2 + a^2(\tau^2 - 4\tau + 2) = 0$. If a zero of (5.5) is not near a pole of the right-hand side, it may be neglected, so that the equation reduces to

$$T\tau^2 - L\tau + \frac{L^2}{4T}\beta = 0, \tag{5.8}$$

which has the solution $\tau = (L/2T)[1 \pm (1-\beta)^{\frac{1}{2}}]$. Hence, the top is unstable if $\beta > 1$ and none of the poles of the right-hand side of (5.5) are near

$$\tau_{\text{pr}} = \frac{L}{2\tau}[1-(1-\beta)^{\frac{1}{2}}] \quad \text{or} \quad \tau_{\text{nu}} = \frac{L}{2T}[1+(1-\beta)^{\frac{1}{2}}]. \tag{5.9}$$

Now let us suppose that a root of (5.5) occurs near a pole $\tau = \tau_0$ of the right-hand side. Near $\tau = \tau_0$, (5.5) reduces to

$$T\tau^2 - L\tau + \frac{L^2}{4T}\beta = \frac{D(\tau_0)}{\tau - \tau_0} + \text{small terms}, \tag{5.10}$$

where $D(\tau_0)$ is a small known parameter, the residue at the pole. Then if

$$T\tau_0^2 - L\tau_0 + \frac{L^2}{4T}\beta \gg D, \tag{5.11}$$

the root in question is

$$\tau = \tau_0 + \frac{D}{T\tau_0^2 - L\tau_0 + L^2\beta/4T},$$

and is real since τ_0 is real.

An exception arises if (5.11) is not satisfied. Suppose, for example, that $\tau_0 \doteqdot \tau_{\text{nu}}$. Then, near $\tau = \tau_{\text{nu}}$, (5.5) reduces to

$$T(\tau_{\text{nu}} - \tau_{\text{pr}})(\tau - \tau_{\text{nu}}) = \frac{D}{\tau - \tau_0}.$$

with real roots only if

$$T(\tau_{\text{nu}} - \tau_{\text{pr}})\left(\frac{\tau_0 - \tau_{\text{nu}}}{2}\right)^2 + D \geqslant 0.$$

Hence, even if $\beta < 1$, the top is unstable if $D < 0$ and

$$|\tau_0 - \tau_{\text{nu}}| \leqslant \left[\frac{-D(\tau_0)}{L(1-\beta)^{\frac{1}{2}}}\right]^{\frac{1}{2}}. \tag{5.12}$$

Similarly, the top is unstable if $D > 0$ and

$$|\tau_0 - \tau_{\text{pr}}| \leqslant \left[\frac{D(\tau_0)}{L(1-\beta)^{\frac{1}{2}}}\right]^{\frac{1}{2}}. \tag{5.13}$$

Thus, if the empty top is stable, it is theoretically possible to render it unstable by introducing a small quantity of liquid. It is also possible to render an unstable top stable if $\beta - 1$ is small, but the conditions are complicated and not of great interest. In most cases the only poles which can lead to instability are those of X_k, Z_k, and the residues at these poles are always real and negative so that only

(5.12) is relevant. A slightly more convenient function than D for practical applications is $R > 0$, where $\rho a^6 R^2 + cD = 0$. Thus, R is a positive real number which depends, like the position of the poles, only on c/a, b/a. The theoretical condition (5.12) may thus be restated as follows. The top is unstable if any pole τ_0 of X_k, Z_k satisfies

$$\frac{|\tau_0 - \tau_{\text{nu}}|}{R}\left(\frac{cL(1-\beta)^{\frac{1}{2}}}{\rho a^6}\right)^{\frac{1}{2}} \leqslant 1. \tag{5.14}$$

In the calculations connected with (5.14), there are actually a double infinite set of poles to be considered, because to each integer j the determinant of the coefficients of X_k, Z_k in (5.7) and (5.8) has an infinite number of zeroes. Fortunately, however, k can be absorbed into c/a, and so it is only necessary to tabulate the poles with $j = 0$. Tables from which the leading poles and the corresponding values of R may be determined for given values of b/a, $c/\{(2j+1)a\}$ have been computed on behalf of the author by Dr Gilles and some of them are displayed here (tables 1–5). In the computation it was assumed that poles in which $\tau_0 < 0$, or $\tau_0 > 0{\cdot}20$ were not of great interest. The way in which the tables are to be used

τ_0	$\frac{c}{(2j+1)a}$	R	$\frac{c}{(2j+1)a}$	R 0·0	$\frac{c}{(2j+1)a}$	R 0·0
0·00	0·995	0·000	0·478	000	0·310	000
0·02	1·018	0·058	0·490	070	0·319	019
0·04	1·042	0·118	0·503	144	0·327	040
0·06	1·066	0·181	0·516	223	0·336	062
0·08	1·091	0·246	0·530	307	0·345	086
0·10	1·117	0·313	0·544	396	0·355	111
0·12	1·144	0·382	0·559	491	0·364	139
0·14	1·172	0·454	0·574	591	0·375	168
0·16	1·201	0·528	0·590	697	0·385	198
0·18	1·231	0·604	0·607	809	0·397	231
0·20	1·262	0·682	0·624	928	0·408	266

TABLES 1–5. Tables from which the leading poles τ_0 of X_k, Z_k and the corresponding residues may be calculated as functions of $c/(2j+1)a$, b/a.

TABLE 1. $b = 0$

τ_0	$\frac{c}{(2j+1)a}$	R	$\frac{c}{(2j+1)a}$	R 0·0	$\frac{c}{(2j+1)a}$	R 0·0
0·00	0·947	0·000	0·387	000	0·224	000
0·02	0·968	0·055	0·398	047	0·230	010
0·04	0·991	0·113	0·408	096	0·236	020
0·06	1·015	0·172	0·419	148	0·242	031
0·08	1·039	0·234	0·430	202	0·249	044
0·10	1·065	0·298	0·442	259	0·256	057
0·12	1·092	0·365	0·454	319	0·263	071
0·14	1·120	0·434	0·467	383	0·271	086
0·16	1·149	0·506	0·480	450	0·279	103
0·18	1·181	0·580	0·494	521	0·287	121
0·20	1·214	0·657	0·509	597	0·295	140

TABLE 2. $b^2/a^2 = 0{\cdot}20$

τ_0	$\frac{c}{(2j+1)a}$	R	$\frac{c}{(2j+1)a}$	R 0·0	$\frac{c}{(2j+1)a}$	R 0·00
0·00	0·842	0·000	0·281	000	0·154	000
0·02	0·861	0·047	0·288	023	0·158	039
0·04	0·881	0·097	0·296	046	0·162	081
0·06	0·901	0·148	0·304	071	0·166	127
0·08	0·923	0·201	0·312	098	0·171	177
0·10	0·945	0·256	0·320	125	0·176	231
0·12	0·969	0·314	0·329	154	0·181	289
0·14	0·994	0·374	0·338	185	0·186	352
0·16	1·020	0·437	0·348	218	0·191	419
0·18	1·048	0·503	0·358	252	0·197	492
0·20	1·077	0·572	0·369	289	0·203	571

TABLE 3. $b^2/a^2 = 0{\cdot}40$

τ_0	$\frac{c}{(2j+1)a}$	R	$\frac{c}{(2j+1)a}$	R 0·0	$\frac{c}{(2j+1)a}$	R 0·00
0·00	0·697	0·000	0·182	0000	0·096	000
0·02	0·712	0·035	0·186	0080	0·098	012
0·04	0·728	0·071	0·191	0164	0·101	025
0·06	0·744	0·109	0·196	0252	0·104	040
0·08	0·762	0·149	0·202	0345	0·107	055
0·10	0·780	0·190	0·207	0444	0·110	072
0·12	0·799	0·233	0·213	0547	0·113	091
0·14	0·819	0·278	0·219	0657	0·116	111
0·16	0·840	0·326	0·225	0773	0·119	132
0·18	0·862	0·375	0·232	0896	0·123	155
0·20	0·886	0·427	0·239	1027	0·126	180

TABLE 4. $b^2/a^2 = 0{\cdot}60$

τ_0	$\frac{c}{(2j+1)a}$	R	$\frac{c}{(2j+1)a}$	R 0·00	$\frac{c}{(2j+1)a}$	R 0·000
0·00	0·496	0·000	0·088	000	0·045	000
0·02	0·506	0·019	0·091	014	0·047	019
0·04	0·517	0·039	0·093	028	0·048	040
0·06	0·529	0·060	0·096	043	0·049	062
0·08	0·541	0·081	0·098	059	0·050	087
0·10	0·553	0·104	0·101	077	0·052	113
0·12	0·566	0·127	0·104	094	0·053	142
0·14	0·580	0·152	0·107	113	0·055	174
0·16	0·594	0·178	0·110	134	0·057	207
0·18	0·609	0·205	0·113	155	0·058	244
0·20	0·625	0·234	0·117	178	0·060	284

TABLE 5. $b^2/a^2 = 0{\cdot}80$

may be exemplified as follows. Suppose that $\tau_{\rm nu} = 0{\cdot}1$, $c/a = 3$, $b^2/a^2 = 0{\cdot}20$. In order to test for instability, we use the table in which $b^2/a^2 = 0{\cdot}20$, and to begin with look for any poles in $0 \leqslant \tau_0 \leqslant 0{\cdot}2$ with $j = 0$ and therefore with $c/\{(2j+1)a\} = 3$. There are none. Next we try $j = 1$ so that $c/\{(2j+1)a\} = 1$.

There is a zero in the first table at $\tau \doteqdot 0{\cdot}047$ and at which $R = 0{\cdot}13$. Thirdly, try $j = 2$, so that $c/\{(2j+1)a\} = 0{\cdot}600$. There are no poles in $0 \leqslant \tau_0 \leqslant 0{\cdot}2$. Further values of j may be considered in the same way. Having found all the relevant τ_0, the modified form of (5.14) may now be used to test for instability.

In the Appendix below, Prof. Ward describes some experiments which he has carried out to check the stability criterion in (5.14). It was observed that there was a range of filling ratios on either side of the filling ratio, which corresponded to the theoretical value of principal mode of instability for which the top was violently unstable. However, this range corresponds to a stability criterion

$$-3{\cdot}9 < \frac{(\tau_0 - \tau_{nu})}{R}\left[\frac{cL(1-\beta)^{\frac{1}{2}}}{\rho a^6}\right]^{\frac{1}{2}} < 2{\cdot}7, \tag{5.15}$$

instead of (5.14) as required by the theory. Possible reasons for this discrepancy are discussed by Ward, but without positive conclusions. In any practical application it may be wiser at this stage to use (5.15) instead of (5.14), but it must be admitted that the supporting evidence is not sufficient to be convincing.

Only one other theoretical mode of instability could be detected with certainty in Ward's experiments, which may possibly have been due to the difficulty of designing an apparatus both strong enough to withstand the large forces at the principal mode of instability and yet sensitive enough to show up the weak secondary modes.

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APPENDIX

BY G. N. WARD

In order to test Prof. Stewartson's theory of the stability of a rotating cylindrical shell which is partially filled with liquid, experiments were made with a constant-speed gyrostat. The rotor had a cavity $1\frac{1}{8}$ in. in diameter and $3\frac{3}{8}$ in. long, and was supported by two ball races mounted in a cage. The rotor was driven through a flexible coupling by a small 3-phase induction motor which was also mounted in the cage, and was fed from a variable frequency alternator, the electrical connexions being through some special woven copper wire of great flexibility. The cage was connected by gymbal bearings to a comparatively massive support, the inner gymbal ring having small lead weights attached so that the equivalent moments of inertia of the system were the same about both gymbal axes. The cage was adjustable in the inner gymbal and was positioned in such a way that the centre of mass of the whole system (without liquid) coincided with the centre of rotation, thus making the system into a gyrostat, and in this state the moment of inertia about the gymbal axes was $8{\cdot}95\,\text{lb. in.}^2$. The equivalent moment of inertia of the rotating parts about the axis of symmetry was inferred from a dynamic experiment in which the frequencies of nutation and rotation were measured by stroboscopic means: the ratio of these frequencies was 0·112, from which it follows that the required equivalent moment of inertia was

$$0{\cdot}112 \times 8{\cdot}95 = 1{\cdot}002\,\text{lb.in.}^2.$$

The frequency of precession was very small, being less than 1 cycle/min at a rotor speed of 6000 r.p.m. with the cavity full of liquid, which was the state of maximum unbalance.

The liquid used for the experiment was a mixture of light lubricating oil and liquid paraffin, which had a kinematic viscosity of 23·9 centistokes at 70 °F and 12·8 centistokes at 100 °F, and 47·5 g of this liquid were required to fill the cavity completely. The experimental procedure was to weigh the rotor plus any residual liquid, determine the mass of the residual liquid by subtracting the weight of the empty rotor, and then add the liquid 1 cm^3 at a time, observing the stability or instability after each addition. To make quantitative estimates of the instability, use was made of the fact that a light far up in the laboratory roof was reflected conveniently in the cap of the upper rotor bearings, and that this reflexion appeared to trace out a small circle whose diameter was proportional to the amplitude of the nutational motion. This circle was observed through a slot, cut in thin metal sheet, which was $\frac{1}{8}$ in. wide over half its length and $\frac{1}{4}$ in. wide over the other half, and by moving this slot along its length, and measuring the time taken for the circle to increase in diameter from just filling the narrow portion of the slot to just filling the wider portion, the rate of increase of the amplitude of the oscillation could be determined. This rather crude but effective method of instrumentation proved to be most convenient in use; precautions were taken to ensure that the relative positions of the slot, the gyrostat, and the observer's eye were maintained during an observation. In this way observations of instability were made with a rotor speed of 6000 r.p.m., this being considered to be the highest speed at which it was desirable to run the gyrostat. Even at this speed, the violent instabilities at resonance and in the subsequent tests at large amplitudes, mentioned below, proved to be too much for the gymbal bearings, which suffered some damage, and also for the rotor bearings, which were thrown slightly out of alignment.

The results of the tests are shown in figure 1, where the reciprocal of the time to double the amplitude (for small amplitudes) is plotted against the filling-ratio. In the notation of this paper, when

$$\gamma_n = \frac{\text{nutational frequency}}{\text{rotational frequency}} = 0{\cdot}112,$$

$$n = \text{number of waves radially},$$

$$j + \tfrac{1}{2} = \text{number of waves axially},$$

the theoretical filling-ratios $(1-b^2/a^2)$ for resonance are shown in table 6.

It will be seen from figure 1 that the main resonance $(n,j) = (1,1)$ at filling-ratio 0·66, and the second resonance (1, 2) at filling-ratio 0·23 can be detected with certainty, and that the experimental values of the filling-ratio agree well with the theoretical predictions. Other minor resonances appear to occur, but their positions do not agree with any of the filling-ratios in table 6, and it is possible that these resonances may be spurious. However, there is no doubt about the existence of the main resonance at filling-ratio 0·66, and the results at and near this resonance can be used to rest the theoretical predictions for the range of filling-ratios in which instability occurs.

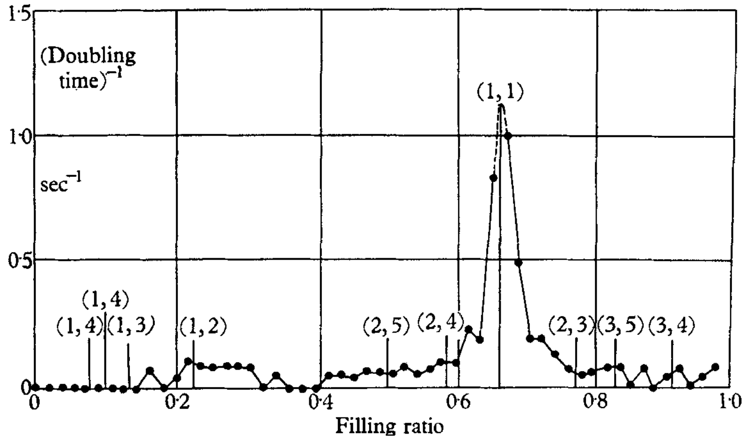

FIGURE 1. Experimental results. Filling ratio against reciprocal of time (in seconds) for amplitude to be doubled, at rotor speed of 6000 r.p.m.

	n		
j	1	2	3
0	—	—	—
1	0·66	—	—
2	0·23	—	—
3	0·14	0·78	—
4	0·10	0·60	0·92
5	0·08	0·50	0·83

TABLE 6

The limits of filling-ratio for instability are very vague in figure 1, but it seems that the gyrostat is definitely unstable between filling-ratios of 0·63 and 0·70. If we define a number A by

$$A = \frac{(\tau_0 - \tau_{nu})}{R}\left[\frac{cL(1-\beta)^{\frac{1}{2}}}{\rho a^6}\right]^{\frac{1}{2}},$$

then the theoretical criterion given in equation (5.14) becomes $-1 \leqslant A \leqslant 1$ for instability. In the present case, the values of A corresponding to the filling-ratios 0·63 and 0·70 are respectively 2·7 and $-3·9$, so the experimental range of A for instability is $-3·9 < A < 2·7$, which is considerably greater than the theoretical range.

It is interesting to speculate on possible reasons for this disagreement. It seems that there are four major factors which would affect the motion and which are neglected in the present theory, namely, the effect of gravity on the liquid filling, the displacement of the axis of rotation from the axis of the cylinder, the effect of non-linear terms, and the effect of friction in the gymbal bearings. As to the effect of gravity, some qualitative experiments at lower rotational speeds (4000 and 5000 r.p.m.) showed that it was appreciable, but the ranges of instability were smaller at the lower rotational speeds, so the effect will not provide an explanation for the discrepancy between the theory and experiment. The effect of the second factor should be small provided that displacement of the axis of symmetry is kept small, and in the experiments the axis of symmetry never made an initial angle of more than about 3° with the axis of nutation (which was approximately vertical in all tests), so the resultant axis of rotation never made an initial angle of more than about 0·3° with the axis of symmetry; nevertheless, the extra inertia forces might have been significant, and to test for this, some large initial displacements in nutation were applied. It was found that the range of instability was increased by doing this, and in fact the gyrostat could be made unstable for almost all filling-ratios tested by giving it a sufficiently violent initial impulse. Unfortunately these tests have not been made systematically owing to the bearing failures described above. The effects of non-linearities are difficult to estimate and must play their part in the increased instability at large displacements noted above; it is difficult to believe that they are of great importance for the small displacements used in the quantitative tests. The fourth possibility is that friction in the gymbal bearings added an appreciable amount of instability to the system. The gymbal bearings were hard conical pivots bearing on ball races; although such bearings leave something to be desired from the frictional point of view, they were adopted to withstand the large forces which were anticipated at the main resonance, and even then they were damaged. Friction in these bearings would be expected to be de-stabilizing, and the fact that the oscillations were never damped at any filling-ratio might lend support to the view that friction there was the cause of the disagreement between the theoretical and experimental ranges of instability; however, the theory indicates that the liquid filling should never give rise to damping of the oscillations, and the results for small filling-ratios show that any de-stabilization caused by bearing friction must have been very small and unlikely to produce the comparatively large effects under discussion.

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